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A Minimum Cost Flow Algorithm in a Disjunctive Network

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Abstract

A trackwork scheduling problem of a network is considered, and a new algorithm is proposed. The railway is divided into several sections, in each of which trackwork must be done once a day. The problem is to choose trackwork time in each section such that the total amount of waiting time of passengers is the minimum.

In order to describe the movement of passengers, the time-distance plane is dissected into small parallelograms, and the number of passengers in each area is considered. These quantities are regarded as flows in a network, each arc of which has its capacity and cost, and a method to construct the network is shown.

Under one choice of trackwork time, the flow of passengers is computed by the well-known minimum cost flow algorithm. By enumerating all cases, we could obtain the optimum choice. In the proposed branch and bound algorithm, however, only the promising cases are generated so that the computation time is greatly decreased.

1. *Introduction*

Let us consider a railway of about 300 - 700 kilometers. The railway is divided into several *sections*, in each of which *trackwork* (i.e.

maintenance of the railway) must be done once a day. Then, the problem we have is to choose trackwork time in each section such that the total amount of waiting time of passengers is the minimum.

However, the diagram of limited express trains is fixed. We can only manage to change the diagram of slow trains. Therefore, trackwork must be scheduled in a time interval between two limited express trains with slight adjustment of slow trains. If the trackwork could not be scheduled adequately, smooth transportation would not be achieved.

In order to describe the flow (or movement) of passengers by slow trains, we dissect the time-distance plane (i.e. the plane where the diagram is to be drawn) into small areas (parallelograms) and consider the following quantities:

The number of passengers by slow trains through each area;

The number of passengers who must wait at the stations because of trackwork.

Also, values of the following quantities are given:

The number of passengers who arrive at stations from outside the railway;

The number of passengers who leave the railway.

Thus, we have a network flow of passengers. Each arc of the network has its capacity and cost (cost per unit flow), where the former stands for the upper bound of the number of passengers and the latter for the waiting time. The way of dissecting the time-distance plane and that of constructing the network are shown in 2.

In the present paper we assume that passengers travel such that the total amount of their waiting time is the minimum. Under this assumption. if we choose one trackwork time for each section, the flow of the passengers is computed the well known *minimum cost flow algorithm*. By enumerating all possible combinations of trackwork time and applying the minimum cost flow algorithm to each of them, we can find the optimum combination. The number of the combinations are, however, large. In the algorithm proposed in 3 only the promising combinations are generated so that the coputation time is greatly decreased.

2. *Scheduling trackwork and a disjunctive network*

In the following our railway is double-tracked, and trackwork is scheduled in each track separately. Therefore, we discuss the problem for one track only.

Let the railway be divided into p sections $\sigma_1, \sigma_2, \ldots, \sigma_p$ by *main stations* $A_0, A_1, A_2, \ldots, A_{p-1}, A_p$. (There are, of course, many other stations --- *small stations* --- in each section.) Each section σ_i starts at A_{i-1} and ends at A_i. Also, let there be d limited express trains $L_1, L_2, \ldots, L_d$.

In order to describe the flow of passengers by slow trains, we dissect the time-distance plane π by the following lines or curves:

Horizontal (i.e. parallel to time axis) lines $a_0, a_1, a_2, \ldots, a_{p-1}$, a corresponding to $A_0, A_1, A_2, \ldots, A_{p-1}, A_p$.

Piecewise linear curves $\ell_1, \ell_2, \ldots, \ell_d$ corresponding to the diagram of limited express trains $L_1, L_2, \ldots, L_d$. We denote the cross point of a_i and ℓ_j by P_{ij}.

Lines s_{ij}'s passing the point P_{ij} of gradient of slow trains. The line corresponds to the slow train which starts the station A_i at the same time as the limited express train L_j (if such a slow train exists). We denote the cross point of s_{ij} with a_{i+1} by $Q_{i+1,j}$, with a_{i-1} by $R_{i-1,j}$.

Thus, we have the following areas (parallelograms) of two kinds:

Waiting area. Parallelograms $P_{ij}Q_{i+1,j}P_{i+1,j}R_{ij}$'s. Any slow train in such an area is outstripped by a limited express train at a small station;

Nonwaiting area. Parallelograms $R_{ij}P_{i+1,j}Q_{i+1,j-1}P_{i,j-1}$'s. Any slow train in such an area is not outstripped by a limited express train at a small station.

The time-distance plane π is dissected into these areas, where we identify the time t+24 with t (fig. 1).

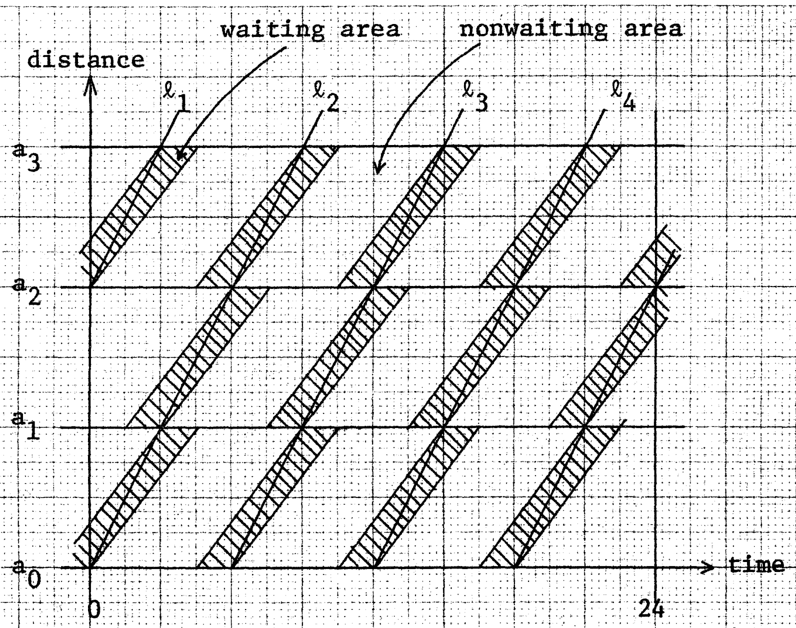

Fig. 1. Dissection of the time-distance plane π.

In the following we assume that the flow (or movement) of passengers is uniform in each area, respectively. Then, the flow of passengers is described by the following quantities.

1° The number of passengers that travel by slow trains through each waiting area (fig. 2). This value is bounded by the number of small stations for wait in the section. This upper bound is

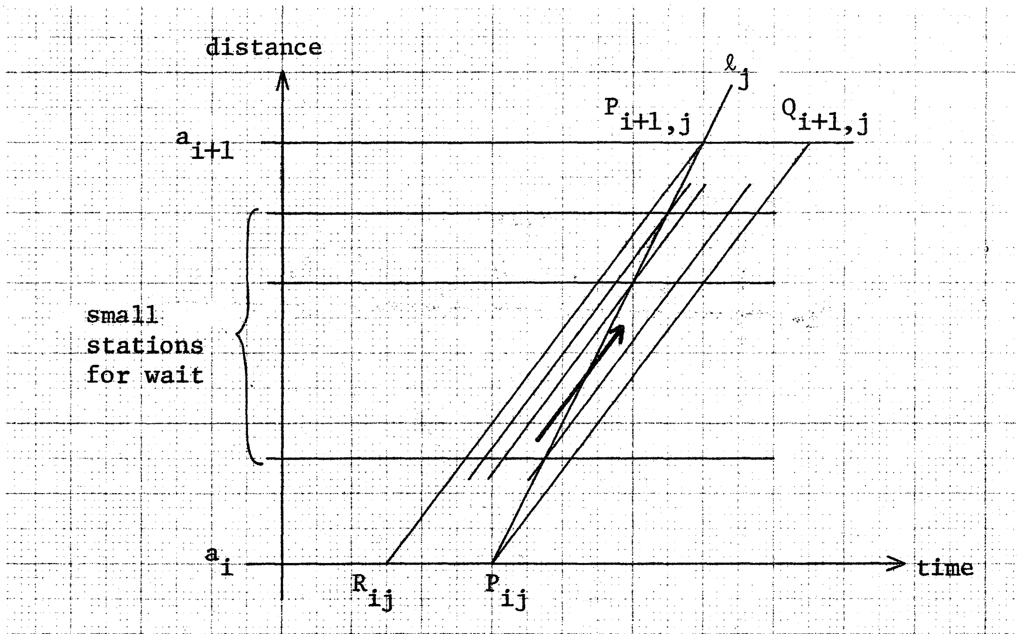

Fig. 2. A waiting area.

the *capacity* for this quantity. Passengers must wait a certain time at a small station. This waiting time is the *cost* for this quantity.

2° The number os passengers that travel by slow trains through each nonwaiting area (fig. 3). Let this area be $R_{ij}P_{i+1,j}Q_{i+1,j-1}P_{i,j-1}$. This value is bounded by $P_{i,j-1}R_{ij}/g$, where g is the minimum time gap that must be between two trains. The bound $P_{i,j-1}R_{ij}/g$ is the capacity for this quantity. The cost is equal to zero.

3° The number of passengers that come into each area from outside the time-distance plane π, i.e. the passengers that arrive at stations from outside the railway (fig. 4). Since we assume that the flow of passengers is uniform in each area, we must only consider passengers that arrive at main stations. The value of this quantity is given.

4° The number of passengers that go outside the time-distance

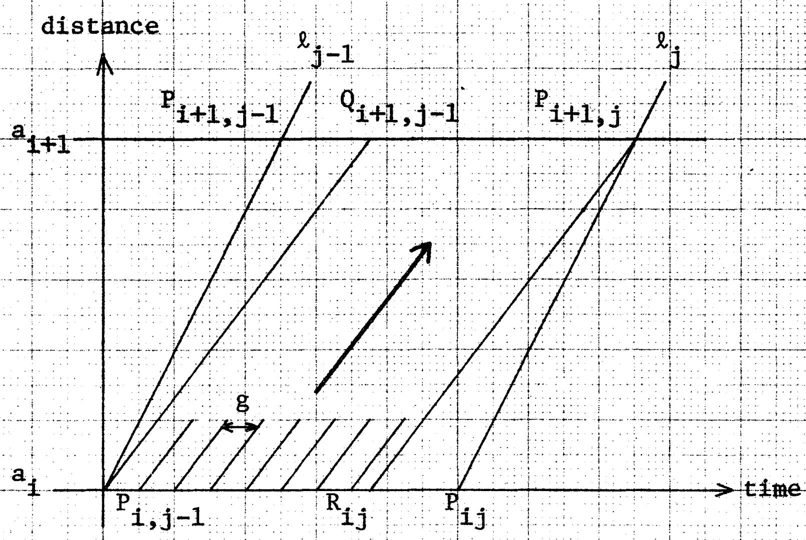

Fig. 3. A nonwaiting area.

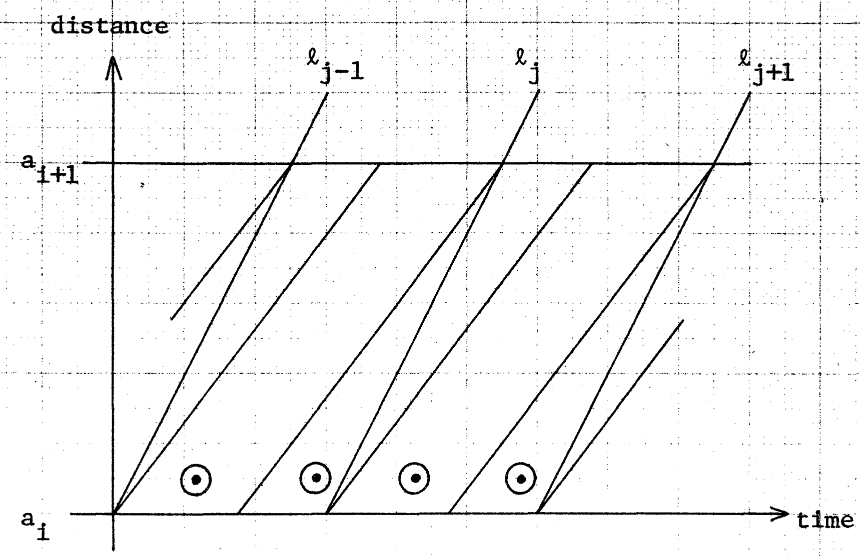

Fig. 4. Passengers that arrive at station a_i from outside the railway.

plane π from each area, i.e. the passengers that leave the railway (fig. 5). By the same reason as in 3° we must only consider passengers that leave main stations. The value of these quantities vary under choices of trackwork time (therefore, unknown variable), but the total amount for each section, i.e. the number of passengers that leave each main station in a day, is given.

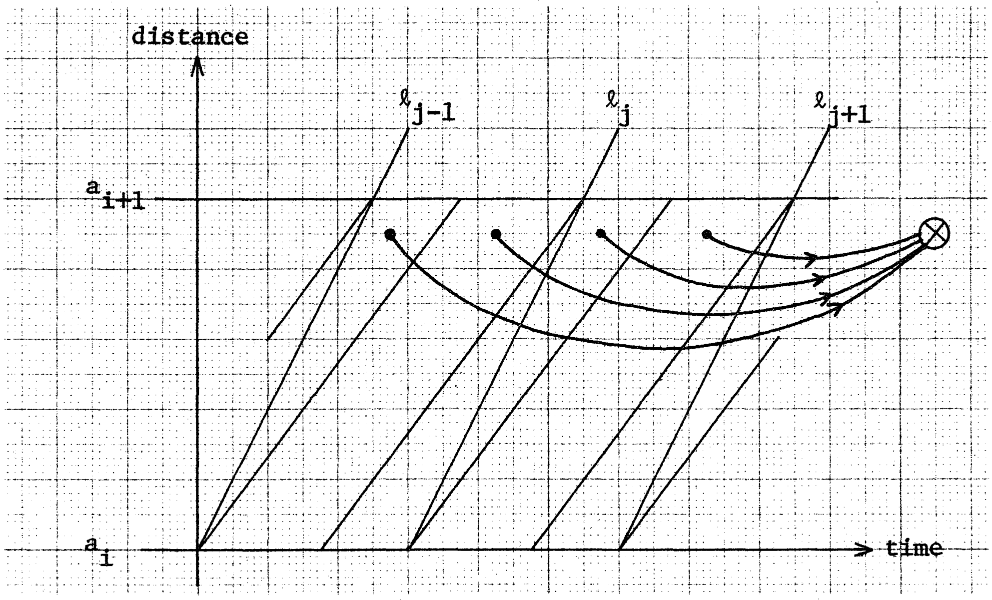

Fig. 5. Passengers that leave the railway.

5° The number of passengers that move from each area to the adjacent one in the positive direction of the distance axis, i.e. the passengers that travel from a section to the next one (fig. 6). Let two areas α and α′ be adjacent and let the length of the common edge of α and α′ be L. The capacity for this quantity is L/g, where g has been defined in 2°, and the cost is zero.

6° The number of passengers that move from each area to the adjacent one in the positive direction of the time axis, i.e. the passengers that must wait at main stations because of trackwork (fig. 7). Let two areas β and β′ be adjacent and let the sum of the length of

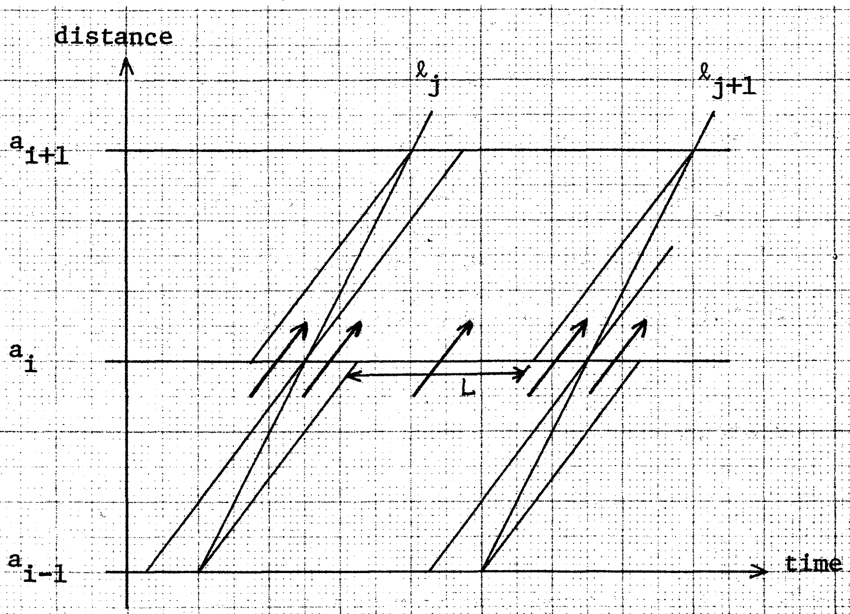

Fig. 6. Passengers that travel to the next section.

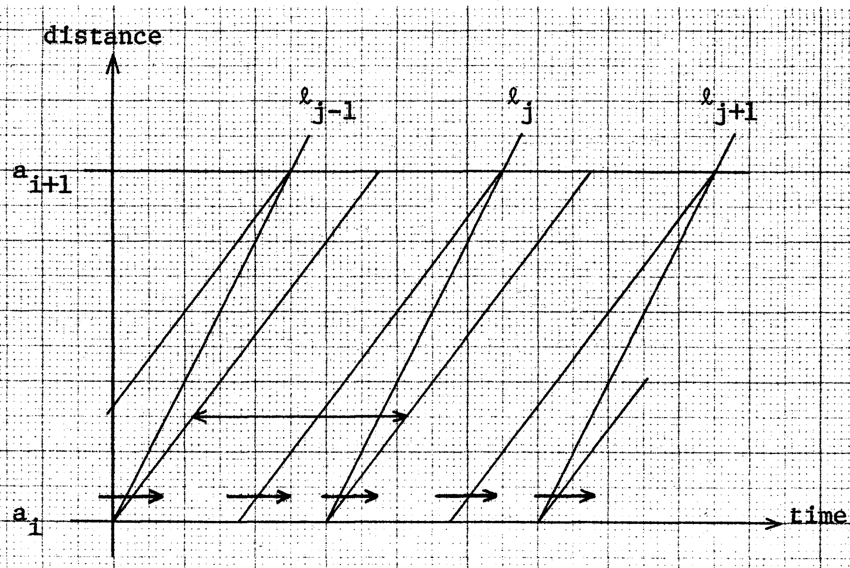

Fig. 7. Passengers that must wait because of trackwork.

two edges in the direction of time axis of β and β' be L'. The cost for this quantity is $L'/2$, and the capacity is infinity.

These quantities are interpreted as flows in a network constructed as follows.

(1) In every area we place two vertices --- an *origin* and a *destination* --- and a directed arc from the former to the latter. Flows in these arcs correspond to the quantities of 1° and 2°. The origin of each area is a *source*. The divergence of each source, which corresponds to the quantity of 3°, is given.

(2) For every section we place a vertex called *sink*. From the destination in each area to the sink of the section we connect a directed arc. Flows in these arcs correspond to the quantities of 4°, and the capacity and cost are the same as in 4°. The value of a sink is the total number of passengers that leave the section in a day, which is given.

(3) From the destination of every area we connect a directed arc to the origin of the adjacent area of the next section. Flows in these arcs correspond to the quantity of 5°, and the capacity and the cost are the same as in 5°.

(4) From the origin of every area we connect a directed arc to the origin of the next area of the same section. Flows in these arcs correspond to the quantity of 6°, and the capacity and the cost are the same as in 6°.

An example of such a network is shown in fig. 8.

As mentioned in 1, trackwork is to be done in a time interval between two limited express trains. Therefore, trackwork time in a section consists of three areas: one nonwaiting area and the adjacent two waiting areas (fig.9). Let the candidates of trackwork time (i.e. sets of three

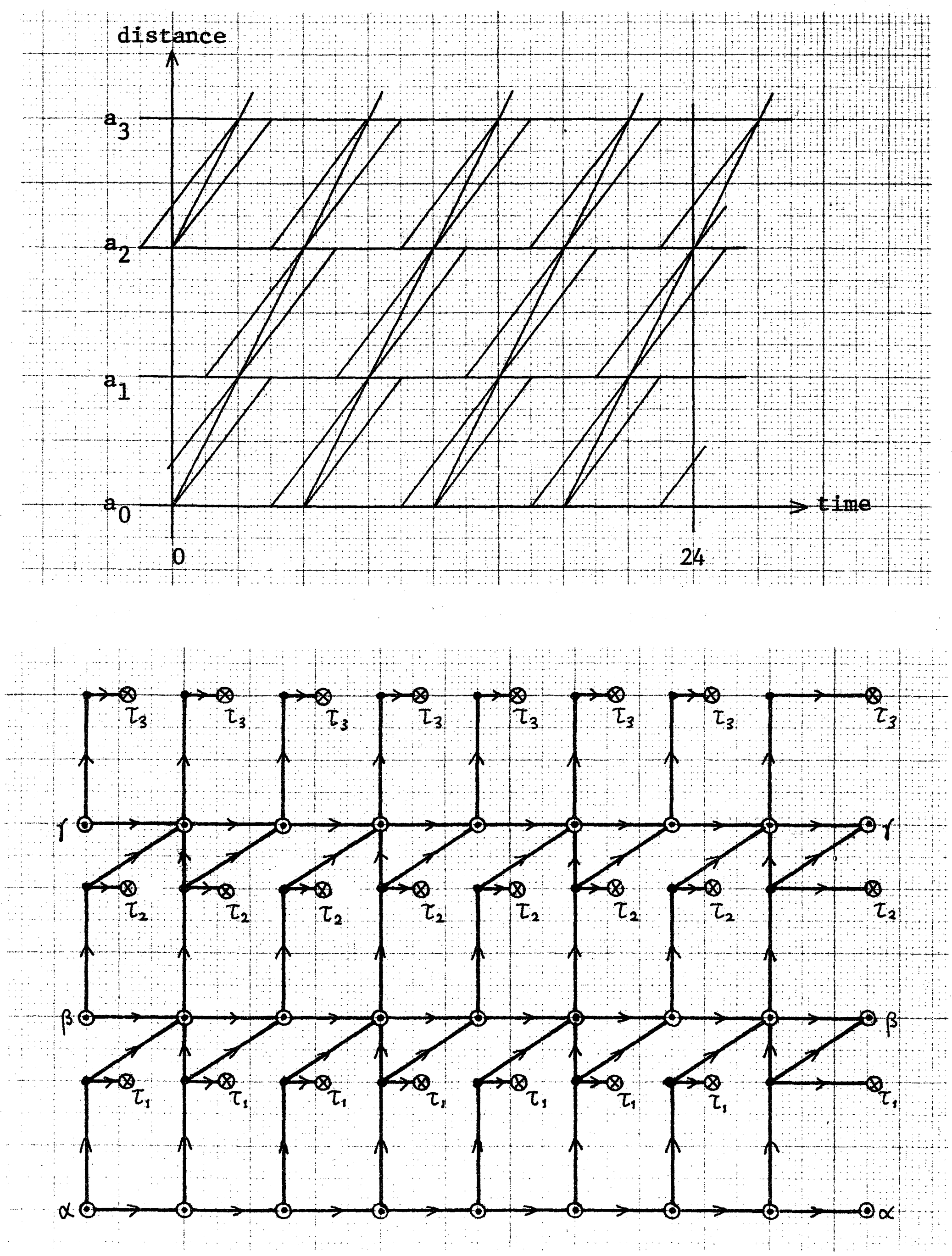

Fig. 8. Dissection of the time-distance plane and the corresponding network.

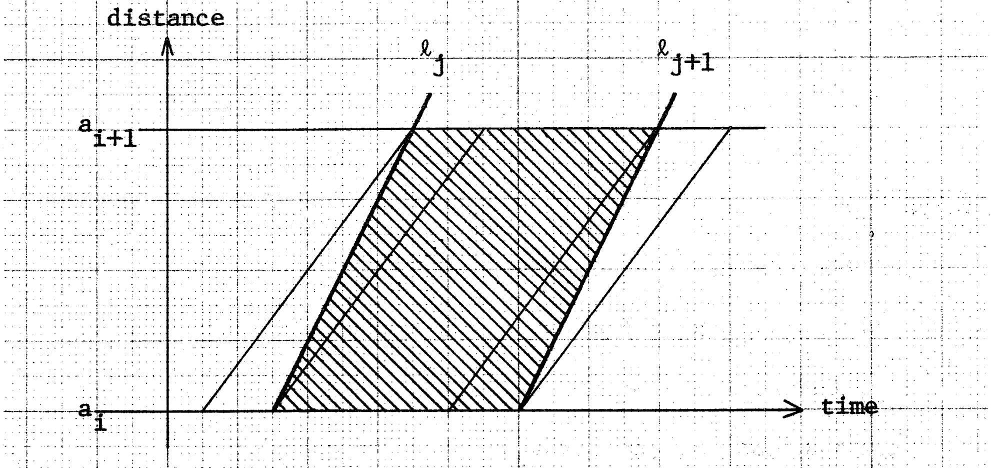

Fig. 9. A trackwork time for a section.

areas) in the section σ_i be $D_i(1)$, $D_i(2), \ldots, D_i(d_i)$. If candidates $D_1(\ell_1)$, $D_2(\ell_2), \ldots, D_p(\ell_p)$ are chosen for sections $\sigma_1, \sigma_2, \ldots, \sigma_p$, respectively, corresponding 3p arcs (see 1° and 2°) must be deleted from the network. We denote such a subnetwork by $N(\ell_1, \ell_2, \ldots, \ell_p)$. We have $d_1 \times d_2 \times \cdots \times d_p$ subnetworks.

For every subnetwork $N(\ell_1, \ell_2, \ldots, \ell_p)$ we denote by $\Phi(\ell_1, \ell_2, \ldots, \ell_p)$ the total amount of waiting time of passengers by slow trains. In the present paper we assume that passengers travel such that the total amount of waiting time is the minimum. Under this assumption, the value of $\Phi(\ell_1, \ell_2, \ldots, \ell_p)$ is computed the well known minimum cost flow algorithm [1] for $N(\ell_1, \ell_2, \ldots, \ell_p)$. Our aim is to obtain $\hat{\ell}_1, \hat{\ell}_2, \ldots, \hat{\ell}_p$ such that

$$\Phi(\hat{\ell}_1, \hat{\ell}_2, \ldots, \hat{\ell}_p) = \min_{\ell_1, \ell_2, \ldots, \ell_p} \Phi(\ell_1, \ell_2, \ldots, \ell_p) \tag{1}$$

The minimum cost flow problem for $N(\ell_1, \ell_2, \ldots, \ell_p)$ is formulated as follows:

Minimize $\qquad z = \sum_{k \in A} e_k x_k$

subject to

$$\sum_{k\in\delta^+ i} x_k - \sum_{k\in\delta^- i} x_k = \begin{cases} r_i & \text{(if vertex } v_i \text{ is a source)}, \\ -t_i & \text{(if vertex } v_i \text{ is a sink)}, \\ 0 & \text{(otherwise)}, \end{cases}$$

$$0 \leqq x_k \leqq c_k \qquad \text{for any } k \in A,$$

where

A: the set of arcs in $N(\ell_1, \ell_2,\ldots, \ell_p)$,

$\delta^+ i$ [$\delta^- i$] : the set of arcs out [into] the vertex v_i,

c_k: the capacity of the arc a_k ,

e_k: the cost of the arc a_k,

r_i: the value of divergence of the source v_i,

t_i: the value of sink v_i,

x_k: the flow in the arc a_k (x_k's are unknown variables).

We denote this problem by $P(\ell_1, \ell_2,\ldots, \ell_p)$.

3. *A branch and bound algorithm*

For a subset F of $\{1,\ldots, p\}$ let us define

$$\tilde{\Phi}(\ell_1,\ldots, \ell_p; F) = \min \{\Phi(\ell'_1,\ldots, \ell'_p) \mid \ell'_i = \ell_i \ (i \in F)\} \tag{2}$$

Then, for any i and j ($i < j$; $i, j \notin F$) we have

$$\begin{aligned} &\tilde{\Phi}(\ell_1,\ldots, \ell_p; F) \\ &\quad = \min \Big(\Phi(\ell_1,\ldots, \ell_i,\ldots, \ell_j,\ldots, \ell_p), \\ &\qquad\qquad \min_{\ell'_i \neq \ell_i} \Phi(\ell_1,\ldots, \ell'_i,\ldots, \ell_j,\ldots, \ell_p; F \cup \{i\}), \\ &\qquad\qquad \min_{\ell'_j \neq \ell_j} \Phi(\ell_1,\ldots, \ell_i,\ldots, \ell'_j,\ldots, \ell_p; F \cup \{i,j\}) \Big) \end{aligned} \tag{3}$$

If we start from $\Phi(\ell_1,\ldots, \ell_p; \emptyset)$ and apply (3) recursively (i.e. *branch procedure*), we can enumerate all possibilities. In order to decrease the computation time, we apply the following test (i.e. *bound procedure*):

(1) Δ-*test*. The optimal solution for the dual problem of $P(\ell_1,\dots, \ell_p)$ gives a feasible solution for the dual problem of any $P(\ell_1',\dots, \ell_p')$ and, therefore, gives a lower bound for $\Phi(\ell_1',\dots, \ell_p')$. If this lower bound is not less than the best value already obtained, the computation of $\Phi(\ell_1',\dots, \ell_p')$ may be omitted.

(2) 0-*test*. For $\ell_1,\dots, \ell_p$ and F we construct a network by deleting only those $D_i(\ell_i)$'s such that $i \in F$. We denote this network by $N(\tilde{\ell}_1,\dots, \tilde{\ell}_p)$ where $\tilde{\ell}_i = \ell_i$ if $i \in F$ and $\tilde{\ell}_i = 0$ if $i \notin F$. Then we have a lower bound for $\tilde{\Phi}(\ell_1,\dots, \ell_p; F)$, i.e.

$$\Phi(\tilde{\ell}_1,\dots, \tilde{\ell}_p) \leqq \tilde{\Phi}(\ell_1,\dots, \ell_p; F)$$

If this lower bound is not less than the best value already obtained, the computation of $\tilde{\Phi}(\ell_1,\dots, \ell_p; F)$ may be omitted.

(3) δ-*test*. This test computes a lower bound for the above $\Phi(\tilde{\ell}_1,\dots, \tilde{\ell}_p)$ by the same technique as Δ-test. If this lower bound is not less than the best value already obtained, the computation of $\tilde{\Phi}(\ell_1,\dots, \ell_p; F)$ may be omitted.

(4) μ-*test*. Let us solve $P(0,\dots, \ell_i,\dots, 0)$. The cost of flows corresponding to section σ_i is denoted by $\mu_i(\ell_i)$. Then we have a lower bound for $\tilde{\Phi}(\ell_1,\dots, \ell_p; F)$, i.e.

$$\mu_1(\ell_1) +\cdots+\mu_p(\ell_p) \leqq \tilde{\Phi}(\ell_1,\dots, \ell_p; F)$$

If this lower bound is not less than the best value already obtained, the computation of $\tilde{\Phi}(\ell_1,\dots, \ell_p; F)$ may be omitted.

4. *Computational results*

An example of computation is shown in fig. 10, where $p = 3$ and $d_1 = d_2 = d_3 = 4$. In fig. 10 underlined argument of Φ means that the section is in F. For example, $\Phi(\underline{3}, 1, 1)$ stands for $\Phi(3, 1, 1; \{1\})$. The asterisks show that no further computation is necessary.

The total number of choices in this example is 64, whereas the minimum cost flow algorithm is executed only 17 times.

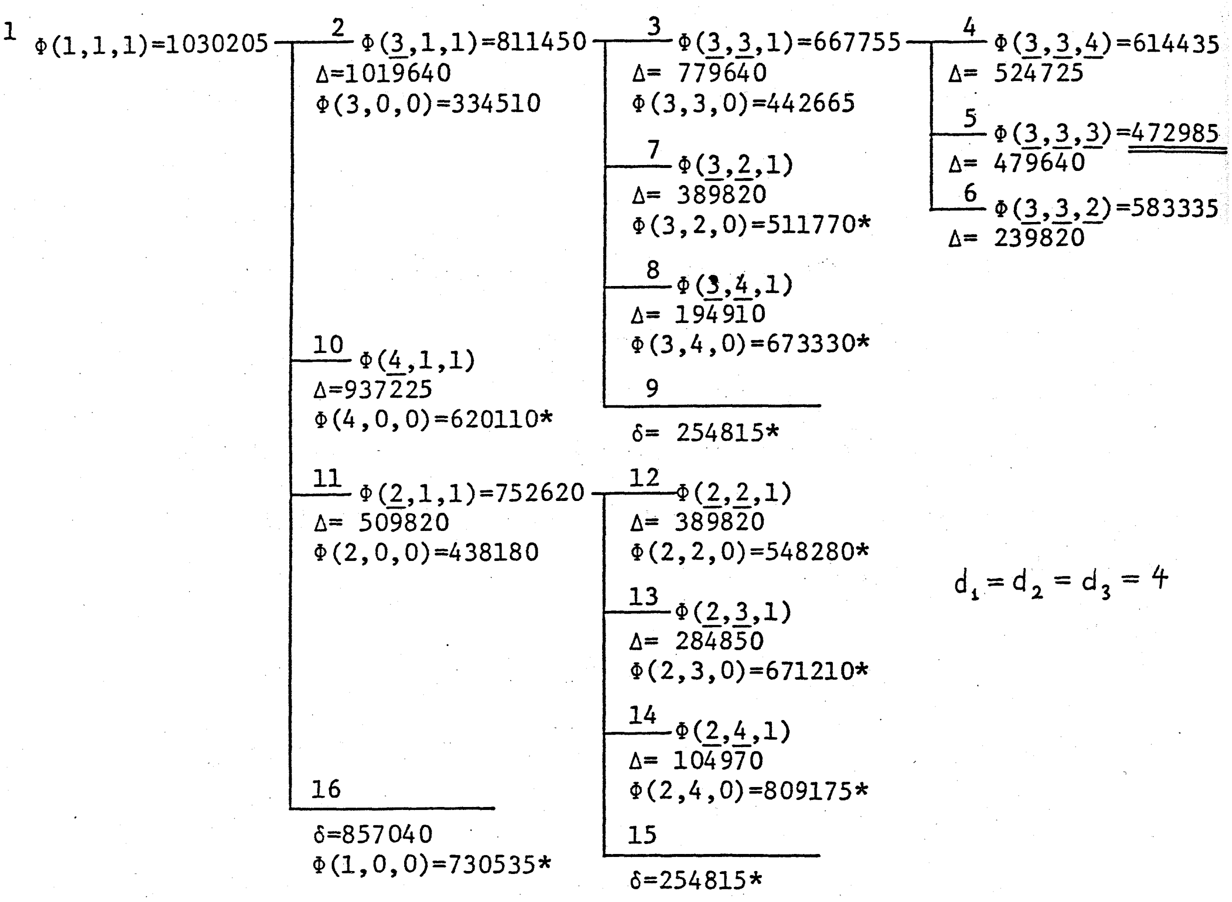

Fig. 10

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